

Combinatorics of free vertex algebras

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INTRODUCTION

This paper illustrates the combinatorial approach to vertex algebra — study of vertex algebras presented by generators and relations. A necessary ingredient of this method is the notion of free vertex algebra. Borcherds [2] was the first to note that free vertex algebras do not exist in general. The reason for this is that vertex algebras do not form a variety of algebras, as defined in e.g. [4], because the locality axiom (see §2 below) is not an identity. However, a certain subcategory of vertex algebras, obtained by restricting the order of locality of generators, has a universal object, which we call the free vertex algebra corresponding to the given locality bound. In [15] some free vertex algebras were constructed and in certain special cases their linear bases were found. In this paper we generalize the construction of [15] and find linear bases of an arbitrary free vertex algebra.

It turns out that free vertex algebras are closely related to the vertex algebras corresponding to integer lattices. The latter algebras play a very important role in different areas of mathematics and physics. They were extensively studied in e.g. [5, 6, 9, 11, 14]. Here we explore the relation between free vertex algebras and lattice vertex algebras in much detail. These results comply with the use of the word “free” in physical literature referring to some elements of lattice vertex algebras, like in “free field”, “free bozon” or “free fermion”.

Among other things, we find a nice presentation of lattice vertex algebras in terms of generators and relations, thus giving an alternative construction of these algebras without using vertex operators. We remark that our construction works in a very general setting; we do not assume the lattice to be positive definite, neither non-degenerate, nor of a finite rank.

Organization of the manuscript

We start with reviewing some basic definitions of vertex and conformal algebra in §1–§3. The reader may consult Kac’s book [11] for more details. Then in §4 we recall the construction of the vertex algebra V_Λ corresponding to an integer lattice Λ , the details can be again found in [11].

In §5 we construct the free vertex algebra $\mathfrak{F}_N(\mathcal{B})$ generated by a set \mathcal{B} so that the order of locality $N(a, b)$ of a pair of generators $a, b \in \mathcal{B}$ is given by an arbitrary symmetric integer-valued function $N : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{Z}$. Theorem 1 describes a set $\mathcal{T} \subset \mathfrak{F}_N(\mathcal{B})$ and claims that \mathcal{T} is a linear basis of $\mathfrak{F}_N(\mathcal{B})$. As a corollary, we find the dimensions of homogeneous components of $\mathfrak{F}_N(\mathcal{B})$. In §11 we will prove that \mathcal{T} spans $\mathfrak{F}_N(\mathcal{B})$, and in §12 we finish the proof of Theorem 1 by showing that \mathcal{T} is linearly independent.

In §6 we construct a vertex algebra homomorphism $\varphi : \mathfrak{F}_N(\mathcal{B}) \rightarrow V_\Lambda$ from the free vertex algebra $\mathfrak{F}_N(\mathcal{B})$ to the vertex algebra corresponding to the lattice $\Lambda = \mathbb{Z}[\mathcal{B}]$. The integer form on Λ is defined by $(a|b) = -N(a, b)$ for $a, b \in \mathcal{B}$. Theorem 2 then states that φ is injective. We will prove this theorem in §12.

In §7, using Theorem 2, we prove a quantitative version of Dong’s lemma. In §8 we apply this lemma to settle a question raised in [16]: we prove that the locality function of a free conformal algebra has quadratic growth.

In §9 we study homogeneous conformal derivations of free vertex algebras. It turns out that a particularly interesting case is when the algebra is generated by a single element a such that $N(a, a) = -1$. In this case $\mathfrak{F}_N(\{a\})$ is embedded into the fermionic vertex algebra $V_{\mathbb{Z}}$. We prove that a homogeneous component of $\varphi(\mathfrak{F}_N(\{a\})) \subset V_{\mathbb{Z}}$ is an irreducible lowest weight module over certain conformal algebra $\widehat{\mathfrak{M}} \subset V_0$, such that the coefficient algebra of $\widehat{\mathfrak{M}}$ is a central extension of the Lie algebra of differential operators on the circle, see [7, 11, 17].

Finally, in §10 we find a presentation of lattice vertex algebras in terms of generators and relations, see Theorem 4. It turns out that the required relations are rather minimal. Our proof is completely combinatorial. The first step is to determine the structure of the vertex algebra in question as a module over the Heisenberg algebra.

Notations

All algebras and linear spaces are over a field \mathbb{k} of characteristic 0. Here are some shortcuts used throughout the paper: $\{n \in \mathbb{Z} \mid n \geq 0\} = \mathbb{Z}_+$, $\frac{1}{k!}n(n-1)\cdots(n-k+1) = \binom{n}{k}$, $\frac{1}{k!}D^k = D^{(k)}$, $Z(L)$ is the center of a Lie algebra L .

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1. FIELDS AND LOCALITY

Let $V = V^{\bar{0}} \oplus V^{\bar{1}}$ be a vector superspace over \mathbb{k} . Take a formal variable z and consider the space $F(V) = F(V)^{\bar{0}} \oplus F(V)^{\bar{1}} \subset \text{End}(V, V((z)))$ of *fields* on V , given by

$$F(V)^p = \left\{ \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \left| p(a(n)) = p, \forall v \in V, a(n)v = 0 \text{ for } n \gg 0 \right. \right\},$$

Here $p(x) \in \mathbb{Z}/2\mathbb{Z}$ is the parity of x . Denote by $\mathbf{1} \in F(V)^{\bar{0}}$ the identity operator, such that $\mathbf{1}(-1) = \text{Id}_V$, all other coefficients are 0.

Let $\iota_{w,z}(w-z)^n$ and $\iota_{z,w}(w-z)^n$ be two different expansions of $(w-z)^n$ into formal power series in the variables w and z :

$$\begin{aligned} \iota_{w,z}(w-z)^n &= \sum_{i \geq 0} (-1)^{n+i} \binom{n}{i} w^{n-i} z^i \in \mathbb{k}[[w, w^{-1}, z]], \\ \iota_{z,w}(w-z)^n &= \sum_{i \geq 0} (-1)^i \binom{n}{i} w^i z^{n-i} \in \mathbb{k}[[w, z, z^{-1}]]. \end{aligned}$$

Of course, if $n \geq 0$ then $\iota_{w,z}(w-z)^n = \iota_{z,w}(w-z)^n$.

Let $a, b \in F(V)$. We say that a is *local* to b if there is some $N \in \mathbb{Z}$ such that

$$a(w)b(z) \iota_{w,z}(w-z)^N - (-1)^{p(a)p(b)} b(z)a(w) \iota_{z,w}(w-z)^N = 0. \quad (1)$$

The minimal $N = N(a, b)$ with this property is called the *order of locality* of a and b . In terms of the coefficients locality means that the following identities hold for all $m, n \in \mathbb{Z}$:

$$\begin{aligned} & \sum_{s \geq 0} (-1)^s \binom{N}{s} a(m-s)b(n+s) \\ & - (-1)^{p(a)p(b)} \sum_{s \leq N} (-1)^s \binom{N}{N-s} b(n+s)a(m-s) = 0. \end{aligned} \quad (2)$$

Usually the locality is defined only for a nonnegative order N [6, 11], but we will need this property in a bigger generality.

For an integer n define a bilinear product $\boxed{n} : F(V) \otimes F(V) \rightarrow F(V)$ by

$$(a \boxed{n} b)(z) = \text{Res}_w \left(a(w)b(z) \iota_{w,z}(w-z)^n - (-1)^{p(a)p(b)} b(z)a(w) \iota_{z,w}(w-z)^n \right). \quad (3)$$

Here Res_w stands for the coefficient of w^{-1} . Explicitly this means

$$(a \boxed{n} b)(m) = \sum_{s \geq 0} (-1)^s \binom{n}{s} a(n-s)b(m+s) - (-1)^{p(a)p(b)} \sum_{s \leq n} (-1)^s \binom{n}{n-s} b(m+s)a(n-s). \quad (4)$$

Clearly, if a and b are local of order N then $a \boxed{n} b = 0$ for $n \geq N$.

If $n = -1$, then $a \boxed{-1} b = :ab: = a_- b + b a_+$ is the so-called *normally ordered product*, where

$$a_{\pm}(z) = \sum_{n \gtrless 0} a(n) z^{-n-1}.$$

In general, if $n < 0$ then

$$\alpha \boxed{n} \beta = : (D^{(-n-1)} \alpha) \beta :,$$

where $D\alpha(z) = \frac{d}{dz}\alpha(z) = \alpha(z) \boxed{-2} \mathbf{1}$.

We remark that the following identities hold:

$$(Da) \boxed{n} b = -n a \boxed{n-1} b, \quad a \boxed{n} (Db) = n a \boxed{n-1} b + D(a \boxed{n} b),$$

in particular, D is a derivation of the products \boxed{n} .

The Dong's lemma [6, 11, 13] states that if $a, b, c \in F(V)$ are three pairwise local fields, then for every $n \in \mathbb{Z}$ the fields $a \boxed{n} b$ and c are local as well. In §7 we will prove a quantitative version of this lemma.

A subspace $\mathfrak{A} \subset F(V)$ such that

- (i) any two fields $a, b \in \mathfrak{A}$ are local,
- (ii) $\mathfrak{A} \boxed{n} \mathfrak{A} \subseteq \mathfrak{A}$ for all $n \in \mathbb{Z}$ and
- (iii) $\mathbf{1} \in \mathfrak{A}$

is called a vertex algebra. By the Dong's lemma, given a collection $\mathcal{S} \subset F(V)$ of pairwise local fields, the closure of $\mathcal{S} \cup \{\mathbf{1}\}$ under the products is a vertex algebra in $F(V)$.

2. AXIOMATIC DEFINITION OF VERTEX ALGEBRAS

Vertex algebras can also be defined axiomatically as follows [11]. Let $\mathfrak{A} = \mathfrak{A}^0 \oplus \mathfrak{A}^1$ be a linear superspace endowed with a sequence of even bilinear operations $\boxed{n} : \mathfrak{A} \otimes \mathfrak{A} \rightarrow \mathfrak{A}$, $n \in \mathbb{Z}$, and a distinguished element $\mathbf{1} \in \mathfrak{A}^0$. Let $D : \mathfrak{A} \rightarrow \mathfrak{A}$ be an even linear map given by $Da = a \boxed{-2} \mathbf{1}$. Consider the *left regular action map* $Y : \mathfrak{A} \rightarrow \mathfrak{A}[[z, z^{-1}]]$ defined by $Y(a)(z) = \sum_{n \in \mathbb{Z}} (a \boxed{n} \cdot) z^{-n-1}$. Then \mathfrak{A} is a vertex algebra if it satisfies the following conditions for any $a, b \in \mathfrak{A}$ and $n \in \mathbb{Z}$:

V1. $a \boxed{n} b = 0$ for $n \gg 0$.

V2. $\mathbf{1} \boxed{n} a = \delta_{n,-1} a$, $a \boxed{n} \mathbf{1} = \begin{cases} 0 & \text{if } n \geq 0, \\ D^{(-n-1)} a & \text{if } n < 0. \end{cases}$

V3. $(Da) \boxed{n} b = -n a \boxed{n-1} b$, $a \boxed{n} (Db) = n a \boxed{n-1} b + D(a \boxed{n} b)$.

V4. The series $Y(a), Y(b) \in \mathfrak{A}[[z, z^{-1}]]$ are local.

This is not a minimal set of axioms, in fact conditions V2 and V3 can be weakened. Note that V3 implies that $Y(Da) = \frac{d}{dz} Y(a)$. For other axiomatic definitions of vertex algebras see [2, 6, 8, 9].

A homomorphism of two vertex algebras \mathfrak{A} and \mathfrak{B} is a map $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ such that $\phi(\mathbf{1}) = \mathbf{1}$ and $\phi(a \boxed{n} b) = \phi(a) \boxed{n} \phi(b)$ for all $n \in \mathbb{Z}$, in particular $\phi(Da) = \frac{d}{dz} \phi(a)$. A module over a vertex algebra \mathfrak{A} is a linear space V such that there is a vertex algebra homomorphism $\phi : \mathfrak{A} \rightarrow F(V)$ of \mathfrak{A} into a vertex subalgebra of $F(V)$. Similarly one defines the notions of ideals, quotients, isomorphisms, etc.

One of the most important properties of vertex algebras is that the left regular action map is in fact an isomorphism of a vertex algebra \mathfrak{A} with a vertex subalgebra $Y(\mathfrak{A}) \subset F(\mathfrak{A})$. In other words, the left regular action map is a faithful representation of \mathfrak{A} on itself. In view of (4) this implies the following identity:

$$\begin{aligned} (a \boxed{n} b) \boxed{m} c &= \sum_{s \geq 0} (-1)^s \binom{n}{s} a \boxed{n-s} (b \boxed{m+s} c) \\ &\quad - (-1)^{p(a)p(b)} \sum_{s \leq n} (-1)^s \binom{n}{n-s} b \boxed{m+s} (a \boxed{n-s} c) \end{aligned} \tag{5}$$

for all $m, n \in \mathbb{Z}$, $a, b, c \in \mathfrak{A}$. An equivalent form of this identity is

$$a \boxed{m} (b \boxed{n} c) - b \boxed{n} (a \boxed{m} c) = \sum_{s \geq 0} \binom{m}{s} (a \boxed{s} b) \boxed{m+n-s} c. \quad (6)$$

Another important identity is the following so-called quasisymmetry identity:

$$a \boxed{n} b = - \sum_{i \geq 0} (-1)^{n+i} D^{(i)} (b \boxed{n+i} a). \quad (7)$$

There is a couple of additional axioms which are often imposed on vertex algebras, see [6, 8, 9, 13]. A vertex algebra \mathfrak{A} is called a *vertex operator algebra* or a *conformal vertex algebra* if

V5. $\mathfrak{A} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{A}_i$ is graded so that $\mathfrak{A}^0 = \sum_{i \in \mathbb{Z}} \mathfrak{A}_i$, $\mathfrak{A}^1 = \sum_{i \in \mathbb{Z} + \frac{1}{2}} \mathfrak{A}_i$, $\mathbb{1} \in \mathfrak{A}_0$ and $\mathfrak{A}_i \boxed{n} \mathfrak{A}_j \subset \mathfrak{A}_{i+j-n-1}$.

V6. There exists an element $\omega \in \mathfrak{A}_2$ such that $N(\omega, \omega) = 4$, $\omega \boxed{0} a = Da$, $\omega \boxed{1} a = (\deg a) a$ for all homogeneous $a \in \mathfrak{A}$, $\omega \boxed{3} \omega = 0$ and $\omega \boxed{4} \omega = c\mathbb{1}$ for some $c \in \mathbb{k}$.

The axiom V6 implies that ω generates the Virasoro conformal algebra $\mathfrak{Vir} \subset \mathfrak{A}$ (see §3 below). The number c is called the *conformal charge* of \mathfrak{A} .

We will call an expression of the form $a_1 \boxed{n_1} \dots \boxed{n_{l-1}} a_l$, $a_i \in \mathfrak{A}$, $n_i \in \mathbb{Z}$, (with an arbitrary order of parentheses) a *vertex monomial* of length l . Using (5) every vertex monomial can be expressed as a linear combination of *right normed* monomials of the form $a_1 \boxed{n_1} (\dots a_{l-2} \boxed{n_{l-2}} (a_{l-1} \boxed{n_{l-1}} a_l) \dots)$. We will call a monomial *minimal* if every its submonomial $u \boxed{n} v$ has the maximal possible value of n , i.e. $n = N(u, v) - 1$.

3. CONFORMAL ALGEBRAS

A conformal algebra is a more general algebraic structure than a vertex algebra, see [11]. The former is obtained from the latter by forgetting about $\mathbb{1}$ and the products \boxed{n} for $n < 0$. The identities V1, V3, (5) and (7) hold in a conformal algebra for $m, n \geq 0$.

Note that if $N \geq 0$ and $n \geq 0$, then (1) and (3) simplify respectively to

$$[a(w), b(z)] (w - z)^N = 0, \quad (8)$$

$$(a \boxed{n} b)(z) = \text{Res}_w [a(w), b(z)] (w - z)^n. \quad (9)$$

This means that the locality for $N \geq 0$ and the products \boxed{n} for $n \geq 0$ both make sense for series $a, b \in L[[z^{\pm 1}]]$ with coefficients in a Lie superalgebra

L . Similar to the case of vertex algebras, a space $\mathfrak{L} \subset L[[z^{\pm 1}]]$ of pairwise local series, such that $D\mathfrak{L} \subseteq \mathfrak{L}$ and $\mathfrak{L} \square_n \mathfrak{L} \subseteq \mathfrak{L}$ for $n \geq 0$, is a conformal algebra and all conformal algebras can be obtained in this way. The Dong's lemma holds for these series as well.

Moreover, for a given conformal algebra \mathfrak{L} one can construct a Lie superalgebra $L = \text{Coeff } \mathfrak{L}$, which is called the *coefficient algebra* of \mathfrak{L} , together with a conformal algebra homomorphism $\mathfrak{L} \rightarrow L[[z^{\pm 1}]]$ which is universal among all representations of \mathfrak{L} by formal series with coefficients in a Lie superalgebra. As a linear space L is equal to $\mathfrak{L} \otimes \mathbb{k}[t^{\pm 1}]$ modulo the linear subspace generated by all the expressions $(Da)(n) + na(n-1)$ for $a \in \mathfrak{L}$ and $n \in \mathbb{Z}$. We denote here $a(n) = a \otimes t^n$.

Examples

Let \mathfrak{g} be a Lie algebra with a symmetric invariant bilinear form $(\cdot | \cdot)$. Consider the corresponding affine Lie algebra $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{k}[t^{\pm 1}] \oplus \mathbb{k}\mathbf{c}$ with the brackets given by

$$[a(m), b(n)] = [a, b](m+n) + \delta_{m,-n} m(a|b) \mathbf{c}, \quad \mathbf{c} \in Z(\widetilde{\mathfrak{g}}).$$

This Lie algebra is the coefficient algebra of the conformal algebra $\mathfrak{G} \subset \widetilde{\mathfrak{g}}[[z^{\pm 1}]]$, generated by the series $\widetilde{a} = \sum_n a(n) z^{-n-1}$ for $a \in \mathfrak{g}$ and $\mathbf{c} = \mathbf{c}(-1)$ so that $D\mathbf{c} = 0$, $N(\widetilde{a}, \widetilde{b}) = 1$ and

$$\widetilde{a} \square \widetilde{b} = \widetilde{[a, b]}, \quad \widetilde{a} \square \mathbf{c} = (a|b) \mathbf{c}.$$

In the case when \mathfrak{g} is an abelian Lie algebra, the corresponding affine algebra $\widetilde{\mathfrak{g}}$ is a Heisenberg algebra, and \mathfrak{G} is a Heisenberg conformal algebra.

Another example of a conformal algebra is obtained from the Virasoro Lie algebra Vir , which is spanned by the elements L_n , $n \in \mathbb{Z}$ and \mathbf{c} with the brackets given by

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n} \binom{m+1}{3} \mathbf{c}, \quad \mathbf{c} \in Z(Vir).$$

It is the coefficient algebra of the conformal algebra $\mathfrak{Vir} \subset Vir[[z^{\pm 1}]]$ generated by the series $\omega = \sum_{n \in \mathbb{Z}} L_{n-1} z^{-n-1}$ and \mathbf{c} . We have $D\mathbf{c} = 0$, $N(\omega, \omega) = 4$ and

$$\omega \square \omega = D\omega, \quad \omega \square \mathbf{1} \omega = 2\omega, \quad \omega \square \mathbf{2} \omega = 0, \quad \omega \square \mathbf{3} \omega = \mathbf{c}.$$

Another example of a conformal algebra will be given in §9.

Conformal operators

Consider first a $\mathbb{k}[D]$ -module V . A *conformal operator* [3] on V is a series $\alpha = \sum_{n \in \mathbb{Z}_+} \alpha(n) z^{-n-1} \in z^{-1}(\text{gl } V)[[z^{-1}]]$ such that $\alpha(n)v = 0$ if $n \gg 0$ for every fixed $v \in V$ and $[D, \alpha] = \frac{d}{dz}\alpha$. For $v \in V$ call $N(\alpha, v) = \min \{n \in \mathbb{Z}_+ \mid \alpha(m)v = 0 \ \forall m \geq n\}$ the order of locality of α and v . Denote by $\text{cgl } V$ the space of all conformal operators on V .

A pair $\alpha, \beta \in \text{cgl } V$ is said to be local of order $N \in \mathbb{Z}_+$ if (8) holds. Also the products $\alpha \boxed{n} \beta \in \text{cgl } \mathfrak{A}$ are defined for $n \geq 0$ by the formula (9). The Dong's lemma also holds for conformal operators. A subspace $\mathfrak{L} \subset \text{cgl } V$ of pairwise local conformal operators, closed under all the products \boxed{n} for $n \in \mathbb{Z}_+$, is a conformal algebra. A module over a conformal algebra \mathfrak{L} is a $\mathbb{k}[D]$ -module V with a conformal algebra homomorphism $\mathfrak{L} \rightarrow \text{cgl } V$.

It is well-known [3] that if $v \in V$ is such that $p(D)v = 0$ for some polynomial $p \in \mathbb{k}[D]$ then $\alpha(n)v = 0$ for every $\alpha \in \text{cgl } V$ and $n \in \mathbb{Z}_+$. In particular, if \mathfrak{A} is a vertex algebra, then $\alpha(n)\mathbf{1} = 0$ for every $\alpha \in \text{cgl } \mathfrak{A}$.

Let \mathfrak{A} be a conformal (in particular, vertex) algebra. A *conformal derivation* [3, 16] is a conformal operator $\alpha \in \text{cgl } \mathfrak{A}$ such that

$$\alpha(m)(a \boxed{n} b) = a \boxed{n} (\alpha(m)b) + \sum_{s \geq 0} \binom{m}{s} (\alpha(s)a) \boxed{m+n-s} b$$

for all $m \in \mathbb{Z}_+$, $n \in \mathbb{Z}$. For example, if $a \in \mathfrak{A}$, then

$$Y(a)_+ = \sum_{n \in \mathbb{Z}_+} Y(a)(n) z^{-n-1}$$

is a conformal derivation of \mathfrak{A} . Denote by $\text{cder } \mathfrak{A} \subset \text{cgl } \mathfrak{A}$ the space of all conformal derivations of \mathfrak{A} . It is not difficult to show that if $\alpha, \beta \in \text{cder } \mathfrak{A}$, then $\alpha \boxed{n} \beta \in \text{cder } \mathfrak{A}$.

Assume that \mathfrak{A} is generated by a set \mathcal{B} . Then a conformal derivation is uniquely defined by its action on \mathcal{B} . It is easy to show that if for $\alpha, \beta \in \text{cder } \mathfrak{A}$ the orders of locality $N(\alpha, \mathcal{B})$, $N(\beta, \mathcal{B})$ are uniformly bounded on \mathcal{B} , then α and β are local.

4. LATTICE VERTEX ALGEBRAS

In this section we give a very important example of vertex algebras — the algebra V_Λ corresponding to an integer lattice Λ . We mostly follow [11], see also [5, 6, 9]. Note, however, that we do not assume that Λ is of a finite rank.

Denote the integer-valued symmetric bilinear form on Λ by $(\cdot | \cdot)$. Let us extend the form to $\mathfrak{h} = \Lambda \otimes_{\mathbb{Z}} \mathbb{K}$. Let $H = \tilde{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{K}[t, t^{-1}] \oplus \mathbb{K}\mathbf{c}$ be the corresponding Heisenberg algebra, see §3.

For every $a \in \Lambda$ consider the canonical relation level 1 representation V_a of H , that is, the irreducible Verma H -module generated by the vacuum vector v_a such that $h(n)v_a = 0$ for $n > 0$, $h(0) = (h|a) \text{Id}$, and $\mathbf{c} = \text{Id}$, see [10]. It is well-known (see e.g. [11]) that the module V_0 has a structure of vertex algebra, such that $v_0 = \mathbb{1}$. The map $\tilde{h} \mapsto h(-1)v_0$ is an injective conformal algebra homomorphism of the conformal Heisenberg algebra \mathfrak{H} to V_0 . From now on we will identify \mathfrak{H} with its image in V_0 . In fact one can show that V_0 is a unique vertex algebra generated by \mathfrak{H} .

For $a \neq 0$, the map $\tilde{h} \mapsto \sum_{n \in \mathbb{Z}} h(n)z^{-n-1} \in F(V_a)$ is a conformal algebra homomorphism of \mathfrak{H} into $F(V_a)$, which extends to the vertex algebra homomorphism $V_0 \rightarrow F(V_a)$, so that V_a becomes a module over the vertex algebra V_0 . Define the Fock space

$$V_{\Lambda} = \bigoplus_{a \in \Lambda} V_a \cong V_0 \otimes \mathbb{K}[\Lambda].$$

Let $\varepsilon : \Lambda \times \Lambda \rightarrow \{\pm 1\}$ be a bimultiplicative map such that

$$\varepsilon(a, b) = (-1)^{(a|a)(b|b)} (-1)^{(a|b)} \varepsilon(b, a). \quad (10)$$

for any $a, b \in \Lambda$. We remark that it is enough to check the identity (10) only when a and b belong to some \mathbb{Z} -basis of Λ ; then (10) will follow for general a, b by bimultiplicativity. The cocycle ε defines an element of $H^2(\Lambda, \{\pm 1\})$.

The main result [5, 6, 9, 11] is that the vertex algebra structure on V_0 can be uniquely extended to V_{Λ} such that $\tilde{h} \boxed{a} v = h(n)v$ for any $h \in \mathfrak{h}$ and $v \in V_{\Lambda}$. The locality of a pair $v_a, v_b \in V$ of vacuum vectors is $N(v_a, v_b) = -(a|b)$ and the products are

$$v_a \boxed{-(a|b)-k-1} v_b = \varepsilon(a, b) (D - b(-1))^{(k)} v_{a+b}, \quad k \geq 0. \quad (11)$$

In particular,

$$v_a \boxed{-(a|b)-1} v_b = \varepsilon(a, b) v_{a+b}, \quad v_a \boxed{-(a|b)-2} v_b = \varepsilon(a, b) a(-1)v_{a+b}.$$

Taking $b = -a$ we get

$$v_a \boxed{(a|a)-1} v_{-a} = \varepsilon(a, a) \mathbb{1}, \quad v_a \boxed{(a|a)-2} v_{-a} = \varepsilon(a, a) \tilde{a}.$$

It follows that the vacuum vectors $v_{\pm a}$, for a running over an integer basis of Λ , generate the vertex algebra V_{Λ} , and (11) defines the vertex algebra

structure on V_Λ uniquely. The vertex algebra V is simple if and only if the form $(\cdot | \cdot)$ is non-degenerate.

The standard way to construct vertex algebra V_Λ is by showing that the vacuum vectors v_a act under the left regular map $Y : V_\Lambda \rightarrow F(V_\Lambda)$ by the so-called vertex operators. In §10 we will construct V_Λ by a different method.

Besides the grading by the lattice Λ , the vertex algebra V_Λ has a grading by $\frac{1}{2}\mathbb{Z}$, so that $\deg v_a = \frac{1}{2}(a|a)$, $\deg \tilde{h} = 1$ for every $h \in \mathfrak{h}$, $\deg \boxed{n} = -n - 1$ and $\deg D = 1$, in particular the axiom V5 always holds. We have decomposition

$$V_a = \bigoplus_{d \in \frac{1}{2}(a|a) + \mathbb{Z}_+} V_{a,d}, \quad V_{a,d} \boxed{n} V_{a',d'} \subset V_{a+a', d+d'-n-1}.$$

We will refer to the grading by Λ as the grading by weights, and write $\text{wt } u = a \in \Lambda$ for $u \in V_a$.

Consider the case when Λ is non-degenerate and of a finite rank l . Let a_1, \dots, a_l and b_1, \dots, b_l be dual bases of \mathfrak{h} , i.e. such that $(a_i | b_j) = \delta_{ij}$. Then the element $\omega = \frac{1}{2} \sum_{i=1}^l a_i \boxed{-1} b_i \in V_0$ generates a copy of the Virasoro conformal algebra \mathfrak{Vir} so that V6 is satisfied, and hence V_Λ becomes a vertex operator algebra.

Remark. The correspondence $\Lambda \mapsto V_\Lambda$ is not a functor, because there is a certain degree of freedom in choosing the cocycle ε , satisfying (10). However, all the resulting vertex algebras V_Λ are isomorphic.

5. FREE VERTEX ALGEBRAS

In this section we construct another example of vertex algebras — a free vertex algebra. Take some set of generators \mathcal{B} . We will assume that \mathcal{B} is linearly ordered. Suppose we have a symmetric function $N : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{Z}$. Consider the category $\mathfrak{Ver}_N(\mathcal{B})$ of vertex algebras generated by the set \mathcal{B} such that in any vertex algebra $\mathfrak{A} \in \mathfrak{Ver}_N(\mathcal{B})$ one has $a \boxed{n} b = 0$ for any $a, b \in \mathcal{B}$ whenever $n \geq N(a, b)$. We set the parity of $a \in \mathcal{B}$ to be $p(a) \equiv N(a, a) \pmod{2}$. The morphisms of $\mathfrak{Ver}_N(\mathcal{B})$ are, naturally, vertex algebra homomorphisms $\phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ such that $\phi(a) = a$ for any $a \in \mathcal{B}$. Then it is easy to see that $\mathfrak{Ver}_N(\mathcal{B})$ has a unique universal object $\mathfrak{F}_N(\mathcal{B})$, called *the free vertex algebra generated by \mathcal{B} with respect to locality bound N* . We construct $\mathfrak{F}_N(\mathcal{B})$ explicitly below.

Remark. It follows easily from the quasisymmetry identity (7) that if $a \in \mathfrak{A}^0$ is an even element of a vertex algebra \mathfrak{A} , then $N(a, a) \in 2\mathbb{Z}$.

Let \mathfrak{A} be a vertex algebra generated by a set \mathcal{B} . For $a, b \in \mathcal{B}$ let

$$N(a, b) = \min \{ n \in \mathbb{Z} \mid a \overline{[m]} b = 0 \quad \forall m \geq n \}$$

be the order of locality. Then there is a surjective homomorphism $\psi : \mathfrak{F}_N(\mathcal{B}) \rightarrow \mathfrak{A}$. Let $\mathcal{R} \subset \mathfrak{F}_N(\mathcal{B})$ be a set of generators of $\text{Ker } \psi$. Then we say that \mathfrak{A} is presented by generators \mathcal{B} , locality bound $N : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{Z}$ and relations \mathcal{R} .

We construct the free vertex algebra $\mathfrak{F}_N(\mathcal{B})$ as follows. Consider the set $\mathcal{X} = \{a(n) \mid a \in \mathcal{B}, n \in \mathbb{Z}\}$ and let $\mathbb{k}[\mathcal{X}]$ be the free associative algebra generated by \mathcal{X} . A module M over $\mathbb{k}[\mathcal{X}]$ is called *restricted* if for any $a \in \mathcal{B}$ and $x \in M$ we have $a(n)x = 0$ for $n \gg 0$. Let $\overline{\mathbb{k}[\mathcal{X}]}$ be the completion of $\mathbb{k}[\mathcal{X}]$ with respect to the topology in which any series $\sum_i p_i$, $p_i \in \mathbb{k}[\mathcal{X}]$, that makes sense as an operator on every restricted module, converge. Let $I \subset \overline{\mathbb{k}[\mathcal{X}]}$ be the ideal generated by the locality relations (2) for all $a, b \in \mathcal{B}$, $N = N(a, b)$. Denote $U = \overline{\mathbb{k}[\mathcal{X}]} / I$. Let $\mathcal{X}_+ = \{a(n) \mid a \in \mathcal{B}, n \geq 0\} \subset \mathcal{X}$ and consider the quotient $U/U\mathcal{X}_+$. This is a restricted left U -module, therefore any $a \in \mathcal{B}$ corresponds to a field $\sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in F(U/U\mathcal{X}_+)$. These fields are local by the construction and hence they generate a vertex algebra $\mathfrak{F}_N(\mathcal{B}) \subset F(U/U\mathcal{X}_+)$, which clearly satisfies all the criteria for being the desired free vertex algebra. By the Goddard-Kac existence theorem [11, Theorem 4.5], the map $\phi \mapsto \phi(-1)1$ is a one-to-one correspondence between $\mathfrak{F}_N(\mathcal{B})$ and $U/U\mathcal{X}_+$, so from now on we will identify the two.

In the special case when $N(a, b) \geq 0$ the locality identities (2) are finite sums of commutators, hence instead of U one can consider an enveloping algebra of a certain Lie algebra. In this case the construction of $\mathfrak{F}_N(\mathcal{B})$ was done in [15]. It was also proved there that if $N \equiv N(a, b) \in 2\mathbb{Z}_+$ is a non-negative even constant, then a linear basis of $\mathfrak{F}_N(\mathcal{B})$ is given by all right normed vertex monomials

$$a_1 \overline{[m_1]} (a_2 \overline{[m_2]} \cdots (a_k \overline{[m_k]} \mathbb{1}) \cdots), \quad a_i \in \mathcal{B}, m_i \in \mathbb{Z}, m_k < 0, \quad (12)$$

such that

$$m_i - m_{i+1} \leq \begin{cases} N & \text{if } a_i \leq a_{i+1}, \\ N-1 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq k-1$. Note that if $m < 0$, then $a \overline{[m]} \mathbb{1} = D^{(-m-1)}a$. In this paper we extend this result to the general case.

Theorem 1. *A linear basis of $\mathfrak{F}_N(\mathcal{B})$ is given by the set of all monomials (12) such that*

$$m_i - m_{i+1} \leq \begin{cases} \sum_{j=i+1}^k N(a_i, a_j) - \sum_{j=i+2}^k N(a_{i+1}, a_j) & \text{if } a_i \leq a_{i+1}, \\ \sum_{j=i+1}^k N(a_i, a_j) - \sum_{j=i+2}^k N(a_{i+1}, a_j) - 1 & \text{otherwise} \end{cases} \quad (13)$$

for $1 \leq i \leq k-1$.

Denote the set of all basic monomials by \mathcal{T} . We will prove this theorem in §11 and §12.

The free vertex algebra $\mathfrak{F} = \mathfrak{F}_N(\mathcal{B})$ is graded by the free abelian semi-group $\mathbb{Z}_+[\mathcal{B}]$, so that the weight of the monomial (12) is $a_1 + \dots + a_k \in \mathbb{Z}_+[\mathcal{B}]$. Let \mathfrak{F}_λ be the space of all elements of \mathfrak{F} of weight $\lambda \in \mathbb{Z}_+[\mathcal{B}]$. We have the decomposition $\mathfrak{F} = \bigoplus_{\lambda \in \mathbb{Z}_+[\mathcal{B}]} \mathfrak{F}_\lambda$. There is also a grading by $\frac{1}{2}\mathbb{Z}$ such that $\deg b = -\frac{1}{2}N(b, b)$ for $b \in \mathcal{B}$. Each homogeneous subspace of weight λ is in its turn decomposed $\mathfrak{F}_\lambda = \bigoplus_{d \in \frac{1}{2}\mathbb{Z}} \mathfrak{F}_{\lambda, d}$, so that

$$\mathfrak{F}_{\lambda, d} \boxed{n} \mathfrak{F}_{\lambda', d'} \subset \mathfrak{F}_{\lambda + \lambda', d + d' - n - 1} \quad \text{and} \quad \mathbf{1} \in \mathfrak{F}_{0, 0}.$$

Note that the parity of a homogeneous element $w \in \mathfrak{F}$ is $p(w) \equiv 2 \deg w \pmod{2}$ so the axiom V5 holds. The basic set \mathcal{T} is homogeneous with respect to both these gradings. Denote $\mathcal{T}_{\lambda, d} = \mathcal{T} \cap \mathfrak{F}_{\lambda, d}$.

Theorem 1 implies the following combinatorial meaning of the dimensions of homogeneous components of $\mathfrak{F}_N(\mathcal{B})$. Fix a weight $\lambda = \sum_{a \in \mathcal{B}} s_a a \in \mathbb{Z}_+[\mathcal{B}]$. There is only one basic monomial $w_{\min}(\lambda) \in \mathcal{T}$ of weight λ that attains the minimal possible degree $d_{\min}(\lambda) = \deg w_{\min}(\lambda)$. It is of the form

$$w_{\min}(\lambda) = a_1 \boxed{m_1} (\dots a_{k-2} \boxed{m_{k-2}} (a_{k-1} \boxed{m_{k-1}} a_k) \dots), \quad (14)$$

where $a_1 \leq a_2 \leq \dots \leq a_k$ and $m_i = \sum_{j=i+1}^k N(a_i, a_j) - 1$ for $i \leq k-1$.

Let $w \in \mathcal{T}$ be a monomial, given by (12). Define a sequence of integers $\eta(w) = (n_1, \dots, n_k) \in \mathbb{Z}^k$ where

$$n_i = \sum_{j=i+1}^k N(a_i, a_j) - 1 - m_j \quad \text{for } 1 \leq i \leq k-1, \quad n_k = -1 - m_k. \quad (15)$$

By (13) we have $n_1 \geq n_2 \geq \dots \geq n_k$ and if $n_i = n_{i+1}$ then $a_i \leq a_{i+1}$. Also, $\sum_i n_i = \deg w - \deg_{\min}(\text{wt } w)$. We can view $\eta(w)$ as a partition of the number $\deg w - \deg_{\min}(\text{wt } w)$ colored by the set \mathcal{B} by setting the color of n_i to be a_i . It is easy to see that this way we get a one-to-one correspondence between $\mathcal{T}_{\lambda,d}$ and the set of corresponding colored partitions of $d - d_{\min}(\lambda)$.

Corollary 1. *The dimension of the homogeneous component $\mathfrak{F}_N(\mathcal{B})_{\lambda,d}$ is equal to the number of \mathcal{B} -colored partitions of $d - d_{\min}(\lambda)$ that contain at most s_a terms of each color $a \in \mathcal{B}$.*

In particular, these dimensions do not depend on the locality bound N , up to a shift.

6. EMBEDDING OF FREE VERTEX ALGEBRAS INTO LATTICE VERTEX ALGEBRAS

The two former sections indicate a striking similarity between lattice vertex algebras and free vertex algebras. In this section we make this similarity precise.

As in §5, let \mathcal{B} be a set and let $N : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{Z}$ be a symmetric function. Let $\Lambda = \mathbb{Z}[\mathcal{B}]$ be the lattice generated by \mathcal{B} , define the bilinear form first for $a, b \in \mathcal{B}$ by $(a|b) = -N(a, b)$ and then extend it to the whole Λ by linearity. By the universality property of the free vertex algebra $\mathfrak{F} = \mathfrak{F}_N(\mathcal{B})$ there is a vertex algebra homomorphism $\varphi : \mathfrak{F} \rightarrow V_\Lambda$ such that $\varphi(a) = v_a$ for each $a \in \mathcal{B}$. Note that φ is homogeneous with respect to the double grading on \mathfrak{F} and V . The following theorem claims that φ is injective.

Theorem 2. *Let $\mathcal{B} \subset \Lambda$ be a linearly independent set. Then the elements v_a for $a \in \mathcal{B}$ generate a free vertex subalgebra in V_Λ .*

We will prove this theorem in §12. Here we deduce the following easy fact from the mere existence of the homomorphism φ and from formula (11). Let $w = w_{\min}(\lambda) \in \mathfrak{F}$ be the basic minimal monomial of weight $\lambda = a_1 + \dots + a_k$ given by (14). Then it is easy to calculate that $\deg w = \deg_{\min}(\text{wt } w) = \frac{1}{2}(\text{wt } w | \text{wt } w)$.

Proposition 1. *Let $w \in \mathfrak{F}_\lambda$ be monomial of weight $\lambda \in \mathbb{Z}_+[\mathcal{B}]$ such that any its submonomial u has degree $\deg u = \deg_{\min}(\text{wt } u) = \frac{1}{2}(\text{wt } u | \text{wt } u)$. Then $\varphi(w) = \pm v_\lambda \neq 0$.*

Proof. Use induction on the length of w . If $w \in \mathcal{B}$, then $\varphi(w) = v_\lambda$ by the definition on φ . Otherwise, $w = u_1 \overline{m} u_2$ and by the induction, $\varphi(u_i) = v_{\text{wt } u_i}$. Since φ is homogeneous, we have $\deg \varphi(w) = \deg w = \frac{1}{2}(\lambda | \lambda)$, hence $\varphi(w) = v_{\text{wt } u_1 + \text{wt } u_2} = v_\lambda$. \square

Note that we do not use either of Theorems 1 or 2 in the proof of Proposition 1.

Clearly the minimal monomial $w_{\min}(\lambda)$ satisfies the assumption of this proposition. It will follow from Theorems 1 and 2 that every such monomial w is minimal (see the end of §2) and is proportional to $w_{\min}(\lambda)$.

7. QUANTITATIVE DONG'S LEMMA

Let again \mathcal{B} be a set with a locality function $N : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{Z}$ and let $\mathfrak{F} = \mathfrak{F}_N(\mathcal{B})$ be the corresponding free vertex algebra.

Lemma 1. *Let $a, b, c \in \mathcal{B}$ and let $n = N(a, b) - k - 1$ for some $k \geq 0$. Then*

$$N(c, b \boxed{n} a) = \begin{cases} N(a, c) + N(b, c) + k & \text{if } N(b, c) > 0 \text{ or} \\ & 0 \leq k \leq -N(b, c), \\ N(a, c) & \text{if } N(b, c) = 0 \text{ or} \\ & 0 < -N(b, c) \leq k. \end{cases} \quad (16)$$

If a, b, c belong to some arbitrary vertex algebra \mathfrak{A} then clearly the locality $N(c, b \boxed{n} a)$ is bounded from above by the right hand side of (16). Note that this estimate applies also to arbitrary fields $a, b, c \in F(V)$, provided they are pairwise local. Indeed, by the qualitative version of Dong's lemma, such fields generate a vertex algebra in $F(V)$.

Proof. By Theorem 2 it is enough to prove the lemma for the case when $a = v_\alpha$, $b = v_\beta$ and $c = v_\gamma$ are vacuum elements in a lattice vertex algebra V_Λ for some vectors $\alpha, \beta, \gamma \in \Lambda$.

Let $m = -(\alpha + \beta | \gamma) + k - j - 1$ for some $j \geq 0$. Using (11) together with the formulas

$$\begin{aligned} v_\beta \boxed{n} (Du) &= n v_\beta \boxed{n-1} u + D(v_\beta \boxed{n} u), \\ v_\beta \boxed{n} (\alpha(-k)u) &= -(\alpha | \beta) v_\beta \boxed{n-k} u + \alpha(-k) (v_\beta \boxed{n} u), \end{aligned}$$

for $\alpha \in \mathfrak{h}$, $\beta \in \Lambda$ and $u \in V_\Lambda$, we obtain

$$\begin{aligned} c \boxed{m} (b \boxed{n} a) &= \varepsilon(\alpha, \beta) \varepsilon(\alpha, \gamma) \varepsilon(\beta, \gamma) \sum_{i=k-j}^k \binom{k - (\beta | \gamma) - j - 1}{i} \\ &\quad \times (D - \alpha(-1))^{(k-i)} (D - (\alpha + \beta)(-1))^{(j-k+i)} v_{\alpha+\beta+\gamma}. \end{aligned}$$

Let j_{\min} be the minimal value of j such that $c[\overline{m}](b[\overline{n}]a) \neq 0$. We have

$$N(c, b[\overline{n}]a) = -(\alpha + \beta|\gamma) + k - j_{\min} = N(a, c) + N(b, c) + k - j_{\min}.$$

It follows that if $(\beta|\gamma) < 0$ or if $(\beta|\gamma) \geq k \geq 0$, then $j_{\min} = 0$; if $(\beta|\gamma) = 0$, then $j_{\min} = k$; finally, if $k > (\beta|\gamma) > 0$, then $j_{\min} = k - (\beta|\gamma)$ and the statement follows. \square

For monomials of length more than three the analogous estimate is more subtle. Note however, that by Corollary 1, if $w \in \mathfrak{F}$ is a vertex monomial in \mathcal{B} such that $\deg w < \deg_{\min}(\text{wt } w)$ then $w = 0$. So we get:

Proposition 2. *Let $w = a_1[\overline{n_1}] \cdots [\overline{n_{l-1}}]a_l \in \mathfrak{F}$, $a_i \in \mathcal{B}$, $n_i \in \mathbb{Z}$, be a vertex monomial. If*

$$\sum_{i=1}^{l-1} n_i > \sum_{1 \leq i < j \leq l} N(a_i, a_j) - l + 1, \quad (17)$$

then $w = 0$.

Proposition 1 shows that sometimes the estimate (17) is the best possible. For $l = 3$ Lemma 1 gives in general a stronger estimate.

8. LOCALITY FUNCTION

Let \mathfrak{L} be a conformal algebra (see §3) generated by a finite set $\mathcal{B} \subset \mathfrak{L}$. In [16] the following integer function was defined. Consider all monomials $w = a_1[\overline{n_1}] \cdots [\overline{n_{l-1}}]a_l \in \mathfrak{L}$, $a_i \in \mathcal{B}$, $n_i \in \mathbb{Z}_+$. Let $S(l)$ be the maximal possible value of $\sum_{i=1}^{l-1} n_i$ such that $w \neq 0$. We call S the locality function of \mathfrak{L} . It depends on the generating set \mathcal{B} , however, if we take another generating set, then the growth of S can change at most by a linear term. It was shown in [16] that if \mathfrak{L} is embeddable into an *associative conformal algebra*, then $S(l)$ must have at most linear growth.

Clearly, similar function can be defined for a finitely generated vertex algebra.

Now let \mathcal{B} be a finite set with a locality bound $N : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{Z}_+$ that takes only non-negative values. Let $\mathfrak{F} = \mathfrak{F}_N(\mathcal{B})$ be the corresponding free vertex algebra (see §5). Let $\mathfrak{L} \subset \mathfrak{F}$ be the conformal algebra generated by \mathcal{B} . It follows from the results of [15] that \mathfrak{L} is a free conformal algebra. It was shown in [16] that if $N \not\equiv 0$ then the locality function $S(l)$ of \mathfrak{L} has at least quadratic growth, and hence \mathfrak{L} cannot be embedded into an associative conformal algebra. It was also conjectured that $S(l)$ has exactly quadratic

growth. Combining Propositions 1 and 2, we see that this conjecture is indeed true.

Corollary 2. *The locality function of the free conformal algebra generated by a set \mathcal{B} with locality bound N is*

$$S(l) = \frac{l(l-1)}{2} \max_{a,b \in \mathcal{B}} N(a,b) - l + 1.$$

9. ACTIONS OF CONFORMAL ALGEBRAS ON FREE VERTEX ALGEBRAS

Return to the setup of §6. Let $\mathfrak{F} = \mathfrak{F}_N(\mathcal{B})$ be the free vertex algebra generated by a set \mathcal{B} with a locality bound $N : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{Z}$. Let $\Lambda = \mathbb{Z}[\mathcal{B}]$ and let $\varphi : \mathfrak{F} \rightarrow V_\Lambda$ be the injective homomorphism provided by Theorem 2, such that $\varphi(a) = v_a$ for $a \in \mathcal{B}$.

We are interested in conformal derivations $\alpha \in \text{cder } \mathfrak{F}$, which are homogeneous of weight 0 and of some degree $m+1 \in \mathbb{Z}$ (see §3). This means that for each $b \in \mathcal{B}$ we have $\alpha(n)b = f_n(b)D^{m-n}b$ for some functions $f_n : \mathcal{B} \rightarrow \mathbb{k}$ when $0 \leq n \leq m$, and $\alpha(n)b = 0$ when $n > m$. Recall that a conformal derivation is uniquely defined by its action on the generators.

We study the case when $m = 0$ in a greater detail, because it will be used later in §12. Recall from §4 that V_0 contains the conformal Heisenberg algebra \mathfrak{H} . This algebra acts on V_Λ by commuting inner conformal derivations \tilde{h}_+ for $h \in \mathfrak{h} = \Lambda \otimes \mathbb{k}$ such that $N(\tilde{h}_+, \tilde{g}_+) = 0$.

Lemma 2. (a) *Let $f : \mathcal{B} \rightarrow \mathbb{k}$ be an arbitrary function. Then there is a unique conformal derivation $\alpha_f \in \text{cder } \mathfrak{F}$ such that $N(\alpha_f, \mathcal{B}) = 1$ and $\alpha_f(0)b = f(b)b$ for all $b \in \mathcal{B}$. For another function $g : \mathcal{B} \rightarrow \mathbb{k}$ we have $N(\alpha_f, \alpha_g) = 0$.*

(b) *Let $h \in \mathfrak{h}$ is such that $f(b) = (h|b)$ for all $b \in \mathcal{B}$. Then $\varphi(\alpha_f(n)x) = h(n)\varphi(x)$ for every $x \in \mathfrak{F}$ and $n \geq 0$.*

Proof. (a). Consider the derivation $\alpha_f(n)$, $n \geq 0$, of the associative algebra $\overline{\mathbb{k}\langle \mathcal{X} \rangle}$ which acts on a generator $b(m) \in \mathcal{X}$ by $\alpha_f(n)(b(m)) = f(b)b(m+n)$, see §5 for the notations. It is easy to see that $\alpha_f(n)$ preserves the ideal $I \subset \overline{\mathbb{k}\langle \mathcal{X} \rangle}$ generated by the locality relations (2), hence $\alpha_f(n)$ is a derivation of the algebra $U = \overline{\mathbb{k}\langle \mathcal{X} \rangle}/I$. Also, we have $[D, \alpha_f(n)] = -n\alpha_f(n-1)$ where D is the derivation of U defined by $Db(m) = -mb(m-1)$. Since $\alpha_f(n)$ can only increase the indices, it preserves the left ideal $U\mathcal{X}_+ \subset U$, therefore $\alpha_f(n)$ acts on the free vertex algebra $\mathfrak{F} = U/U\mathcal{X}_+$.

(b). Both α_f and \tilde{h}_+ are conformal derivations, therefore they are uniquely determined by their action on the generators \mathcal{B} . But for every $b \in \mathcal{B}$ we have $\alpha_f(n)b = \delta_{n,0} (h|b)b$ and $h(n)v_b = \delta_{n,0} (h|b)v_b$, and the claim follows. \square

Let us now consider the case when $m = 1$. A derivation of weight 0 and degree 2 is determined by a pair of functions $f_0, f_1 : \mathcal{B} \rightarrow \mathbb{k}$. One can show that in order for this derivation to be well defined f_0 must be constant. Take $f_0 \equiv 1$ and an arbitrary function $f = f_1 : \mathcal{B} \rightarrow \mathbb{k}$. This defines a conformal derivation $\omega_f \in \text{cder } \mathfrak{F}$ such that $N(\omega_f, \mathcal{B}) = 2$ and $\omega_f(0)b = Db$, $\omega_f(1)b = f(b)b$ for every $b \in \mathcal{B}$. For another function $g : \mathcal{B} \rightarrow \mathbb{k}$ we have $N(\alpha_f, \omega_g) = N(\omega_f, \omega_g) = 2$ and

$$\omega_f \boxed{0} \alpha_g = \frac{d}{dz} \alpha_g, \quad \omega_f \boxed{1} \alpha_g = \alpha_g, \quad \omega_f \boxed{0} \omega_g = \frac{d}{dz} \omega_g, \quad \omega_f \boxed{1} \omega_g = 2\omega_g.$$

In particular each ω_f generates an action of the Virasoro conformal algebra \mathfrak{Vir} . Note that the products $\alpha_f \boxed{m} \omega_g$ can be calculated using (7).

All these statements are proved in a way similar to the proof of Lemma 2. Using Theorem 4 in §10 one can show that the derivation ω_f can be extended from $\varphi(\mathfrak{F})$ to V_Λ if and only if $f(b) = \frac{1}{2}(b|b)$. If V_Λ has the Virasoro element $\omega \in V_0$, then this extension coincides with the inner conformal derivation ω_+ defined by ω .

Remark. This shows that though free vertex algebras are not vertex operator algebras, they have an action of the conformal Virasoro algebra compatible with the $\frac{1}{2}\mathbb{Z}$ -grading.

However, derivations of higher degree exist only in some very special cases.

Bozon-fermion correspondence

Suppose that $\mathcal{B} = \{a\}$ consists of only one element with $N(a, a) = -1$. Then $\Lambda = \mathbb{Z}$ and the form is given by $(m|n) = mn$. The corresponding lattice vertex algebra $V_\mathbb{Z}$ is the main object of the so-called bozon-fermion correspondence, see e.g. [11]. It is well known that the Heisenberg vertex algebra $V_0 \subset V_\mathbb{Z}$ contains a conformal algebra $\widehat{\mathfrak{W}}$ spanned over $\mathbb{k}[D]$ by the elements $p_m = v_{-1} \boxed{-m-1} v_1$, $m \in \mathbb{Z}_+$. The multiplication table is

$$\begin{aligned} p_m \boxed{k} p_n &= \binom{m+n-k}{m} p_{m+n-k} \\ &- \sum_{s=0}^{m-k} (-1)^{k+s} \binom{m+n-k-s}{n} D^{(s)} p_{m+n-k-s} + \delta_{k,m+n+1} (-1)^m \mathbf{1}. \end{aligned}$$

In particular, $p_0 = -\tilde{a}$ generates the Heisenberg conformal algebra $\mathfrak{H} \subset \widehat{\mathfrak{M}}$, and $p_1 = \frac{1}{2}\tilde{a} \boxed{-1} \tilde{a} - \frac{1}{2}D\tilde{a}$ generates the Virasoro conformal algebra $\mathfrak{Vir} \subset \widehat{\mathfrak{M}}$.

The coefficient algebra of $\widehat{\mathfrak{M}}$ is a central extension $\widehat{W} = W \oplus \mathbb{K}\mathbf{c}$ of the Lie algebra $W = \mathbb{K}\langle p, t^{\pm 1} \mid [t, p] = 1 \rangle^{[-]}$ of differential operators on the circle. The subalgebra $W_+ \subset \widehat{W}$ spanned by the coefficients $p_m(n)$ for $n \geq 0$ can be identified with the Lie algebra $\mathbb{K}\langle p, t \mid [t, p] = 1 \rangle^{[-]}$ of differential operators on the disk so that $p_m(n) = \frac{1}{m!}p^m t^n$, see [7, 17].

Using (11) and the fact that $\varepsilon = 1$ we calculate that

$$p_m(n)v_1 = \begin{cases} (-1)^m D^{(m-n)}v_1 & \text{if } m \geq n, \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

Therefore, the inner conformal derivations $(p_m)_+ \in \text{cder } V_{\mathbb{Z}}$ are homogeneous of weight 0 and degree $m+1$. We will now show that $(p_m)_+$ preserves $\varphi(\mathfrak{F})$, so by Theorem 2 it acts on \mathfrak{F} .

Each vacuum vector $v_k \in V_{\mathbb{Z}}$ is a lowest weight vector for the Lie algebra W_+ , meaning that for a homogeneous $g \in W_+$ one has $gv_k = 0$ if $\deg g > 0$ and $gv_k = \lambda(g)v_k$ for some weight $\lambda \in (W_+)_0^*$ if $\deg g = 0$. It is shown in [17] that v_k generates an irreducible lowest weight W_+ -module $U(W_+)v_k \subset V_k$ of the lowest weight λ_k defined by $\lambda_k(p_m(m)) = (-1)^m \binom{m+k}{k}$. It is also proved in [17] that dimension of the homogeneous component of $U(W_+)v_k$ of degree $d + k^2/2$ is equal to the number of partitions of d into at most k terms, i.e. is exactly $\dim \mathfrak{F}_{k, d+k^2/2}$. This motivates the following theorem.

Theorem 3. $\varphi(\mathfrak{F}_k) = U(W_+)v_k$ for every $k \geq 1$.

Proof. Using the dimension argument it is enough to prove only that $U(W_+)v_k \subseteq \varphi(\mathfrak{F}_k)$, though in fact the proof of the other inclusion is no more difficult.

Since $v_k \in \varphi(\mathfrak{F})$, it is enough to show that $W_+\varphi(\mathfrak{F}) \subset \varphi(\mathfrak{F})$. The Lie algebra W_+ is spanned by $p_m(n)$ for $m, n \geq 0$. From (18) it follows that $p_m(n)v_1 \in \varphi(\mathfrak{F})$. Since v_1 is a generator of $\varphi(\mathfrak{F})$, it follows that each $p_m(n)$ preserves $\varphi(\mathfrak{F})$. \square

Remark. One can show that homogeneous conformal derivations of weight 0 and degree $m > 2$ of a free vertex algebra \mathfrak{F} exist if and only if \mathfrak{F} is a tensor product of free vertex algebras, generated by a single generator a such that either $N(a, a) = 0$ or $N(a, a) = -1$.

10. PRESENTATION OF LATTICE VERTEX ALGEBRAS IN TERMS OF GENERATORS AND RELATIONS

In this section we start with a lattice vertex algebra V_Λ corresponding to an integer lattice Λ . Choose a \mathbb{Z} -basis Π of Λ . We may assume that $\varepsilon(a, a) = 1$ for every $a \in \Pi$. By Theorem 2 the vertex subalgebra of V_Λ generated by $\{v_a \mid a \in \Pi\}$ is isomorphic to a free vertex algebra. On the other hand, it follows from (11) that if we complete these generators by v_{-a} 's, then the resulting vertex algebra will be the whole V_Λ .

Theorem 4. *The lattice vertex algebra V_Λ is presented by generators $\{v_a \mid a \in \pm\Pi\}$ with locality bound given by $N(v_a, v_b) = -(a|b)$ and relations $\{v_a \boxed{(a|a)-1} v_{-a} = \mathbf{1} \mid a \in \Pi\}$.*

Note that the quasismmetry identity (7) implies that also $v_{-a} \boxed{(a|a)-1} v_a = \mathbf{1}$ for $a \in \Pi$.

For the proof we need the following lemma.

Lemma 3. *The lattice vertex algebra V_Λ is presented by generators \tilde{a} and $v_{\pm a}$ for $a \in \Pi$ with locality bound given by*

$$N(v_a, v_b) = -(a|b), \quad N(\tilde{a}, \tilde{b}) = 2, \quad N(\tilde{a}, v_b) = 1, \quad (19)$$

and the following set of relations:

- (i) $\tilde{a} \boxed{0} \tilde{b} = 0, \quad \tilde{a} \boxed{1} \tilde{b} = (a|b) \mathbf{1},$
- (ii) $\tilde{a} \boxed{0} v_b = (a|b) v_b,$
- (iii) $v_a \boxed{(a|a)-1} v_{-a} = \mathbf{1},$
- (iv) $Dv_a = \tilde{a} \boxed{-1} v_a,$

for any $a, b \in \pm\Pi$.

The relations (i) mean that \tilde{a} 's generate the Heisenberg vertex algebra $\mathfrak{H} \subset V_\Lambda$. The relations (ii) say that each v_b is a vacuum vector for the representation of the corresponding Heisenberg Lie algebra of weight given by $a \mapsto (a|b)$.

We first show how this lemma implies the theorem.

Proof of Theorem 4. Consider a set of generators $\mathcal{B} = \{X_a, X_{-a} \mid a \in \Pi\}$ with the locality bound $N : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{Z}$ given by $N(X_a, X_b) = -(a|b)$. Let $\mathfrak{F} = \mathfrak{F}_N(\mathcal{B})$ be the corresponding free vertex algebra. For $a \in \pm\Pi$ denote $K_a = X_a \boxed{(a|a)-1} X_{-a}$, $H_a = X_a \boxed{(a|a)-2} X_{-a} \in \mathfrak{F}$. Let $\mathfrak{I} \subset \mathfrak{F}$ be the ideal generated by the relations $K_a = \mathbf{1}$ for $a \in \Pi$. We have to show that the

quotient vertex algebra $\mathfrak{A} = \mathfrak{F}/\mathfrak{I}$ is isomorphic to V_Λ so that $X_a \mapsto v_a$ under the projection map $\pi : \mathfrak{F} \rightarrow \mathfrak{F}/\mathfrak{I} = \mathfrak{A}$.

By Lemma 3 it is enough to show that the elements $\tilde{a} = \pi(H_a)$, $v_a = \pi(X_a) \in \mathfrak{A}$ satisfy the identities (i)–(iv) above. Note that the locality of these elements is already as prescribed by (19), and (iii) holds by the assumption. The identities (i), (ii) and (iv) follow from the following identities that hold in the free vertex algebra \mathfrak{F} for all $a, b \in \pm\Pi$:

$$\begin{aligned} H_a \boxed{k} H_b &= (a|b) K_a \boxed{k-2} K_b, \quad \text{for } k = 0, 1, \\ H_a \boxed{0} X_b &= (a|b) K_a \boxed{-1} X_b, \\ H_a \boxed{-1} X_a &= X_a \boxed{-2} K_a + (a|a) K_a \boxed{-2} X_a. \end{aligned} \tag{20}$$

It is quite easy to prove these identities by straightforward calculations using only the construction of \mathfrak{F} given in §5. However, we can simplify these calculations even further by using Theorem 2. Consider the set $\bar{\Pi} = \{a, \bar{a} \mid a \in \Pi\}$ and extend the symmetric bilinear form to $\bar{\Lambda} = \mathbb{Z}[\bar{\Pi}]$ by $(a|\bar{b}) = -(a|b)$, $(\bar{a}|\bar{b}) = (a|b)$. In the same way we extend the cocycle ε , assuming that $\varepsilon(a, \bar{a}) = \varepsilon(\bar{a}, \bar{a}) = 1$ for $a \in \Pi$. Let $V_{\bar{\Lambda}}$ be the corresponding lattice vertex algebra and let $\varphi : \mathfrak{F} \rightarrow V_{\bar{\Lambda}}$ be the injective homomorphism constructed in §6, such that $\varphi(X_a) = v_a$, $\varphi(X_{-a}) = v_{\bar{a}}$ for $a \in \Pi$. Let us, for example, prove the first identity in (20), the other two are proved in the same way.

Take some $a, b \in \Pi$. By (11) we have $\varphi(K_a) = v_{a+\bar{a}}$ and $\varphi(H_a) = a(-1)v_{a+\bar{a}}$, so, using (5) and (6), we get for any $k \in \mathbb{Z}_+$:

$$\begin{aligned} \varphi(H_a) \boxed{k} \varphi(H_b) &= (a(-1)v_{a+\bar{a}}) \boxed{k} (b(-1)v_{b+\bar{b}}) \\ &= \sum_{s \geq 0} a(-s-1)v_{a+\bar{a}}(k+s)b(-1)v_{b+\bar{b}} \\ &\quad + \sum_{s \geq 0} v_{a+\bar{a}}(k-s-1)a(s)b(-1)v_{b+\bar{b}}. \end{aligned}$$

Since $(b|a+\bar{a}) = (a|b+\bar{b}) = 0$ we have $[b(i), v_{a+\bar{a}}(j)] = a(0)v_{b+\bar{b}} = 0$, so the only non-zero term in the above comes from the second sum when $s = 1$ and we precisely get the first identity in (20). \square

Proof of Lemma 3

Let \mathfrak{A} be the vertex algebra presented by the generators $\{\tilde{a}, v_{\pm a} \mid a \in \Pi\}$ with locality bound (19) and relations (i)–(iv). Since these relations hold in the lattice vertex algebra V_Λ , there is a projection $\pi : \mathfrak{A} \rightarrow V_\Lambda$.

Let $w = v_{b_1} \boxed{n_1} \cdots \boxed{n_{k-1}} v_{b_k} \in \mathfrak{A}$, $b_i \in \pm\Pi$, be a vertex monomial. Denote by $\Xi(w)$ the set of all vertex monomials involving the same indeterminates v_{b_i} 's in the same order as in w and with the same arrangement of parentheses. Let $\mu(w) \in \Xi(w)$ be the unique minimal monomial in $\Xi(w)$, see §2. As in Proposition 1, we have $\pi(\mu(w)) = \pm v_{\text{wt } w} \in V_\Lambda$.

Recall from §4 that V_Λ is a module over the Heisenberg algebra $H = \mathfrak{h} \otimes \mathbb{k}[t, t^{-1}] \oplus \mathbb{k}\mathbf{c}$ for $\mathfrak{h} = \Lambda \otimes_{\mathbb{Z}} \mathbb{k}$. As before we denote $a \otimes t^n = a(n)$. The algebra $H = H_- \oplus H_+$ is decomposed into a direct sum of subalgebras $H_- = \{h(n) \mid h \in \mathfrak{h}, n < 0\} \oplus \mathbb{k}\mathbf{c}$ and $H_+ = \{h(n) \mid h \in \mathfrak{h}, n \geq 0\}$. Let $\hat{H}_- = H_- \oplus \mathbb{k}D$ be the Lie algebra obtained from H_- by adjoining the derivation D acting by $h(n) \mapsto -n h(n-1)$ and $D\mathbf{c} = 0$. Let $\hat{H} = \hat{H}_- \oplus H_+$. Then V_Λ is a module over \hat{H} .

For a linear combination $h = \sum_{a \in \Pi} s_a a \in \mathfrak{h}$, $s_a \in \mathbb{k}$, denote $\tilde{h} = \sum_{a \in \Pi} s_a \tilde{a} \in \mathfrak{A}$. We observe that \mathfrak{A} is also a module over \hat{H} with the action given by $h(n)x = \tilde{h} \boxed{n} x$, and the vertex algebra homomorphism $\pi : \mathfrak{A} \rightarrow V_\Lambda$ is also a \hat{H} -module homomorphism.

Lemma 4. *Let $w = \mu(w) \in \mathfrak{A}$ be a right-normed minimal monomial of weight $\lambda \in \Lambda$. Then*

- (a) *For any $a \in \pm\Pi$, we have $N(v_a, w) = -(a|\lambda)$.*
- (b) *$Dw = \lambda(-1)w$.*

Proof. We prove this lemma by induction on the length l of w . If $l = 1$ then $w = v_\lambda$ and statements (a) and (b) are true by assumption.

Assume $l > 1$. Then $w = v_b \boxed{n} u$ for some other maximal right-normed monomial u of weight $\lambda' = \lambda - b$. By the induction hypothesis, $N(v_b, u) = -(b, \lambda') = n + 1$. Using Lemma 1, we have: $N(v_a, v_b \boxed{n} u) = -(a|b) - (a|\lambda') - (b|\lambda') - n - 1 = -(a|\lambda)$. This proves (a).

In order to prove (b) we first calculate, using (5):

$$\begin{aligned} (b(-1)v_b) \boxed{n} u &= \sum_{s < 0} b(s)(v_b \boxed{n-s-1} u) + \sum_{s \geq 0} v_b \boxed{n-s-1} (b(s)u) \\ &= b(-1)(v_b \boxed{n} u) + (b|\lambda') v_b \boxed{n-1} u. \end{aligned}$$

Now we have, using the induction hypothesis and (6),

$$\begin{aligned} D(v_b \boxed{n} u) &= (b(-1)v_b) \boxed{n} u + v_b \boxed{n} (\lambda'(-1)u) \\ &= b(-1)(v_b \boxed{n} u) + (b|\lambda') v_b \boxed{n-1} u \\ &\quad - (b|\lambda') v_b \boxed{n-1} u + \lambda'(-1)(v_b \boxed{n} u) \\ &= \lambda(-1)(v_b \boxed{n} u). \end{aligned}$$

□

An element $x \in \mathfrak{A}$ is called vacuum, if for every $h \in \mathfrak{h}$ one has $h(n)x = 0$ if $n > 0$, and $h(0)x = \lambda(h)x$ for some $\lambda \in \mathfrak{h}^*$. It is easy to see that a minimal monomial $w = \mu(w) \in \mathfrak{A}$ is a vacuum element. We prove next that the minimal monomials generate \mathfrak{A} over the extended Heisenberg algebra \widehat{H} . It will follow then by the representation theory of Heisenberg algebras (see e.g. [10]) that as a module over \widehat{H} the vertex algebra $\mathfrak{A} = \bigoplus_{w=\mu(w)} U(\widehat{H})w$ is decomposed into a direct sum of irreducible highest weight modules generated by minimal monomials. Recall that by (5) a vertex algebra is spanned by right-normed monomials in its generators.

Lemma 5. *Let $w \in \mathfrak{A}$ be a right-normed monomial in the variables $\{v_a \mid a \in \pm\Pi\}$. Then $w \in U(\widehat{H}_-)\mu(w)$.*

Proof. As before we prove this lemma by induction on the length l of w . If $l = 1$, then $w = \mu(w) = v_a$. Otherwise, $w = v_a \boxed{n} u$ for some right-normed monomial u . By induction, we have $u = g\mu(u)$ for some $g \in U(\widehat{H}_-)$. Applying the formulas

$$\begin{aligned} v_a \boxed{n} (Du) &= n v_a \boxed{n-1} u + D(v_a \boxed{n} u), \\ v_a \boxed{n} (h(-k)u) &= -(a|h) v_a \boxed{n-k} u + h(-k)(v_a \boxed{n} u), \quad h \in \mathfrak{h}, \end{aligned}$$

which follow from V3 and (6), we can express w as

$$w = \sum_{j \in \mathbb{Z}} g_j v_a \boxed{j} \mu(u), \quad g_j \in U(\widehat{H}_-).$$

Thus we have reduced the lemma to the case when $w = v_a \boxed{n} u$ for a minimal monomial $u = \mu(u)$.

Let $\lambda = \text{wt } u \in \Lambda$. By Lemma 4(a), we have $N(v_a, u) = -(a|\lambda)$. Assume that w is not maximal, then $n = -(a|\lambda) - k - 1$ for some $k > 0$. Using Lemma 4(b) and the formulas above, we get

$$\begin{aligned} \lambda(-1)(v_a \boxed{n+1} u) &= (a|\lambda) v_a \boxed{n} u + v_a \boxed{n+1} Du \\ &= (a|\lambda) v_a \boxed{n} u + (n+1) v_a \boxed{n} u + D(v_a \boxed{n+1} u) \\ &= D(v_a \boxed{n+1} u) - k v_a \boxed{n} u. \end{aligned}$$

Therefore, $v_a \boxed{n} u = \frac{1}{k}(D - \lambda(-1))(v_a \boxed{n+1} u)$, and the lemma follows. □

Note that here we already recover the formula (11), up to a sign.

As in the proof of Theorem 4, consider the free vertex algebra $\mathfrak{F} = \mathfrak{F}_N(\mathcal{B})$ generated by the set $\mathcal{B} = \{X_a \mid a \in \pm\Pi\}$. Recall that \mathfrak{F} is graded by the semilattice $\bar{\Lambda}_+ = \mathbb{Z}_+[\bar{\Pi}]$, where $\bar{\Pi} = \{a, \bar{a} \mid a \in \Pi\}$, so that $\text{wt } X_a = a$, $\text{wt } X_{-a} = \bar{a}$ for $a \in \Pi$. Let $\psi : \mathfrak{F} \rightarrow \mathfrak{A}$ be the vertex algebra homomorphism such that $\psi(X_a) = v_a$ for $a \in \pm\Pi$.

By Corollary 1, all monomials in \mathfrak{F} of a weight $\lambda \in \bar{\Lambda}_+$ having the minimal possible degree $d_{\min}(\lambda)$ are proportional to $w_{\min}(\lambda)$, given by (14). If λ contains a pair a, \bar{a} , then by permuting the variables in $w_{\min}(\lambda)$ we obtain a minimal monomial of the form $u \boxed{a} (v_a \boxed{a|a-1} v_{-a}) \in \mathfrak{F}_\lambda$. Applying (iii) we get that $\psi(w_{\min}(\lambda))$ is proportional to $\psi(w_{\min}(\lambda - a - \bar{a}))$ in \mathfrak{A} . It follows that minimal monomials in \mathfrak{A} in the variables $\{v_a \mid a \in \pm\Pi\}$ are parametrized by the lattice Λ . Combining this with Lemma 5, we see that the vertex algebra \mathfrak{A} is decomposed into a direct sum $\mathfrak{A} = \bigoplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ of irreducible highest weight \widehat{H} -modules \mathfrak{A}_λ . Since V_Λ has the same decomposition, the projection $\pi : \mathfrak{A} \rightarrow V_\Lambda$ must be an isomorphism. This finishes the proof of Lemma 3.

Remark. In fact it is not very difficult to make the last argument without a reference to Corollary 1, thus rendering the whole proof of Theorem 4 more or less independent on Theorems 1 and 2.

Remark. One can prove Lemma 3 in a different way, using the construction of V_Λ by vertex operators, and the so-called method of Z -algebras, that originates to the work of Lepowsky and Wilson [12], see also [5, 9, 14]. This remark is due to C. Dong.

11. PROOF OF THEOREM 1, PART I

Let as before \mathcal{B} be a set with a symmetric locality bound $N : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{Z}$ and let $\mathfrak{F} = \mathfrak{F}_N(\mathcal{B})$ be the corresponding free vertex algebra. Let $\mathcal{X} = \{a(n) \mid a \in \mathcal{B}, n \in \mathbb{Z}\}$. As in §5, consider the completion $\overline{\mathbb{k}\langle\mathcal{X}\rangle}$ of the free associative algebra generated by \mathcal{X} . Every element $g \in \overline{\mathbb{k}\langle\mathcal{X}\rangle}$ is a linear combination, possibly infinite, of words \mathcal{X}^* in the alphabet \mathcal{X} .

By §5 there is a map $\rho : \overline{\mathbb{k}\langle\mathcal{X}\rangle} \rightarrow \mathfrak{F}$, defined by $a_1(n_1) \cdots a_k(n_k) \mapsto a_1 \boxed{n_1} (a_2 \boxed{n_2} \cdots (a_k \boxed{n_k} \mathbb{1}) \cdots)$. Under this map the set \mathcal{T} , described by Theorem 1, can be identified with the set of words $a_1(n_1) \cdots a_k(n_k) \in \mathcal{X}^*$ satisfying the condition (13). In this section we prove that $\rho\mathcal{T}$ spans \mathfrak{F} over \mathbb{k} .

Let $w \in \mathcal{X}^*$ and $f \in \overline{\mathbb{k}\langle\mathcal{X}\rangle}$. A rule $w \xrightarrow{r} f$ on $\overline{\mathbb{k}\langle\mathcal{X}\rangle}$ is a partially defined linear map $\overline{\mathbb{k}\langle\mathcal{X}\rangle} \rightarrow \overline{\mathbb{k}\langle\mathcal{X}\rangle}$ which is applicable to an element $g \in \overline{\mathbb{k}\langle\mathcal{X}\rangle}$ if w occurs in the decomposition of g into a linear combination of words from

\mathcal{X}^* . The result h of the application of r to g is obtained by substituting f instead of w ; in this case we write $g \xrightarrow{r} h$. For a set of rules \mathcal{R} denote by $M(\mathcal{R}) = \{w \in \mathcal{X}^* \mid \exists w \longrightarrow f \in \mathcal{R}\}$ the set of words to which at least one rule from \mathcal{R} is applicable.

Now we construct a set of rules \mathcal{R} that will correspond to the locality relations (2) in the generators \mathcal{B} . Let $w = a_1(n_1) \cdots a_k(n_k) \in \mathcal{X}^*$. Define

$$m_j = \sum_{i=j+1}^k N(a_j, a_i) - \sum_{i=j+2}^k N(a_{j+1}, a_i) \quad \text{for } 1 \leq j \leq k-2, \quad (21)$$

$$m_{k-1} = N(a_{k-1}, a_k).$$

Suppose there is $1 \leq j \leq k-1$ such that $n_j - n_{j+1} > m_j$ or $n_j - n_{j+1} = m_j$ and $a_j > a_{j+1}$. Then \mathcal{R} contains the rule

$$w \longrightarrow a_1(n_1) \cdots a_{j-1}(n_{j-1}) f_j a_{j+2}(n_{j+2}) \cdots a_k(n_k)$$

for

$$f_j = - \sum_{s \geq 1} (-1)^s \binom{N}{s} a_j(n_j - s) a_{j+1}(n_{j+1} + s) \\ + (-1)^{p(a_j)p(a_{j+1})} \sum_{s \leq N} (-1)^s \binom{N}{N-s} a_{j+1}(n_{j+1} + s) a_j(n_j - s),$$

where $N = N(a_j, a_{j+1})$.

We need another set \mathcal{Q} of rules on $\overline{\mathbb{k}[\mathcal{X}]}$. Consider a sequence of elements $a_1, \dots, a_k \in \mathfrak{F}$. By the qualitative Dong's lemma there is $S = S(a_1, \dots, a_k) \in \mathbb{Z}$ such that any vertex monomial $w = a_1 \boxed{n_1} \cdots \boxed{n_{k-1}} a_k \in \mathfrak{F}_N(\mathcal{B})$ is equal to 0 whenever $\sum_{i=1}^{k-1} n_i \geq S$.

Remark. It will follow that $S(a_1, \dots, a_k) = \sum_{1 \leq i < j \leq k} N(a_i, a_j) - k + 2$, see Proposition 2. We could prove this right now, but our argument does not make use of this sharp estimate.

Define \mathcal{Q} to be the set of all rules $w \longrightarrow 0$ such that $w = a_1(n_1) \cdots a_k(n_k)$ has the following property: there is $1 \leq j \leq k$ such that $\sum_{i=j}^k n_i \geq S(a_j, \dots, a_k, \mathbf{1})$. In particular, if $n_k \geq 0$, then $w \longrightarrow 0 \in \mathcal{Q}$. We observe that the set \mathcal{T} is exactly the set of all terminal monomials with respect to the rules $\widehat{\mathcal{R}} = \mathcal{R} \cup \mathcal{Q}$, i.e. the set of monomials to which these rules can not be applied. On the other hand, for any rule $w \longrightarrow f \in \widehat{\mathcal{R}}$ the identity $\rho(w - f) = 0$ holds in \mathfrak{F} . Therefore, all we have to prove is that every word $w \in \mathcal{X}^*$ can be reduced to a linear combination of terminal

words by a (possibly, infinite) number of applications of the rules $\widehat{\mathcal{R}}$. This terminal linear combination is necessary finite, because the rules $\widehat{\mathcal{R}}$ preserve weight and degree, and there are only finitely many monomials in \mathcal{T} of given weight and degree, see Corollary 1.

We will apply the rules $\widehat{\mathcal{R}}$ to a word $w \in \mathcal{X}^*$ in such a way that \mathcal{Q} has priority over \mathcal{R} , in other words, we apply \mathcal{R} only if \mathcal{Q} can no longer be applied. Denote by $\Omega(w)$ the set of all words $u \notin M(\mathcal{Q})$ that can appear in the process of applying the rules $\widehat{\mathcal{R}}$ to w . To prove that \mathcal{T} spans \mathfrak{F} it is enough to show the following two things:

- (i) $w \notin \Omega(w)$;
- (ii) $|\Omega(w)| < \infty$.

Indeed, if both (i) and (ii) are true, then the rules \mathcal{R} can be applied to w at most finitely many times, after that w is reduced to a linear combination of \mathcal{T} .

Recall that the set of generators \mathcal{B} is linearly ordered. Extend this order alphabetically to \mathcal{B}^* by comparing the words from left to right. For a word $w = a_1(n_1) \cdots a_k(n_k) \in \mathcal{X}^*$ denote by $w_i = a_i(n_i) \cdots a_k(n_k)$ the i -th tail of w . Let also

$$d(w) = - \sum_{i=1}^k n_i + \sum_{1 \leq i < j \leq k} N(a_i, a_j).$$

Take a word $u = b_1(p_1) \cdots b_k(p_k) \in \Omega(w)$. It is easy to see that (b_1, \dots, b_k) is a permutation of (a_1, \dots, a_k) . Consider two sequences of integers: $(d(w_1), \dots, d(w_k))$ and $(d(u_1), \dots, d(u_k))$. We note that if $d(w) \leq 0$, then $w \in M(\mathcal{Q})$, therefore both claims (i) and (ii) follow from the following statement:

- (iii) For every $1 \leq i \leq k$ we have $d(u_i) \leq d(w_i)$; moreover, if $(d(w_1), \dots, d(w_k)) = (d(u_1), \dots, d(u_k))$, then $b_1 \cdots b_k < a_1 \cdots a_k \in \mathcal{B}^*$.

It is enough to check (iii) for a word u that appears in the right-hand side of a rule $w \rightarrow f \in \mathcal{R}$. Then u is either

$$a_1(n_1) \cdots a_{j-1}(n_{j-1}) a_j(n_j - s) a_{j+1}(n_{j+1} + s) a_{j+2}(n_{j+2}) \cdots a_k(n_k)$$

for $s \geq 1$ or

$$a_1(n_1) \cdots a_{j-1}(n_{j-1}) a_{j+1}(n_{j+1} + s) a_j(n_j - s) a_{j+2}(n_{j+2}) \cdots a_k(n_k)$$

for $s \leq N = N(a_j, a_{j+1})$. In both cases $d(u_i) = d(w_i)$ when $i \neq j + 1$. In the first case we always have $d(u_{j+1}) = d(w_{j+1}) - s < d(w_{j+1})$. In the

second case

$$\begin{aligned} d(u_{j+1}) &= d(w_{j+1}) - \sum_{i=j+2}^k N(a_{j+1}, a_i) + \sum_{i=j+2}^k N(a_j, a_i) + n_{j+1} - n_j + s \\ &\leq d(w_{j+1}) + m_j - n_j + n_{j+1} \end{aligned}$$

where m_j is given by (21). Now since $w \in M(\mathcal{R})$, we have $n_j - n_{j+1} \leq m_j$ and the equality can hold only if $a_j > a_{j-1}$. This proves (iii).

Remark. In fact we have shown that $\widehat{\mathcal{R}}$ is a rewriting system, which is a generalization of the rewriting system constructed in [15] for the case when the locality bound is constant and non-negative. Only minor modifications to the argument in [15] are needed to show that $\widehat{\mathcal{R}}$ is confluent, i.e. the final result of applications of the rules does not depend on the order in which the rules are applied. Then the Diamond lemma [1, 15] would imply that $\rho\mathcal{T}$ is a basis of \mathfrak{F} . However, in §12 we prove that $\rho\mathcal{T}$ is linearly independent in \mathfrak{F} by a different method.

12. PROOF OF THEOREM 1, PART II AND PROOF OF THEOREM 2

Let $\mathfrak{F} = \mathfrak{F}_N(\mathcal{B})$ be a free vertex algebra. Recall that there is a homomorphism $\varphi : \mathfrak{F} \rightarrow V_\Lambda$ for $\Lambda = \mathbb{Z}[\mathcal{B}]$, see §6. Denote also $\Lambda_+ = \mathbb{Z}_+[\mathcal{B}]$. In §11 we have proved that the image of the set $\mathcal{T} \in \mathcal{X}^*$ of words $a_1(n_1) \cdots a_k(n_k) \in \mathcal{X}^*$ satisfying the condition (13) spans \mathfrak{F} over \mathbb{k} under the projection $\rho : \mathbb{k}\langle \mathcal{X} \rangle \rightarrow \mathfrak{F}$. As before we will identify \mathcal{T} with $\rho\mathcal{T}$. In this section we prove that $\varphi\mathcal{T} \subset V_\Lambda$ is linearly independent over \mathbb{k} . Combined with §11 this will prove both Theorem 1 and Theorem 2.

Step 1. First of all we note that without a loss of generality we can assume that the form $(\cdot | \cdot)$ is non-degenerate on Λ . For otherwise we can embed the set \mathcal{B} into a bigger set $\bar{\mathcal{B}}$ preserving the locality bound N , such that the matrix $\{N(a, b)\}_{a, b \in \bar{\mathcal{B}}}$ is non-degenerate. Then, if we have proved the statement for the non-degenerate case, we know that the set \mathcal{T} in linearly independent in $V_{\mathbb{Z}[\bar{\mathcal{B}}]}$, hence it is linearly independent in V_Λ .

Step 2. Recall from §9 that for each function $f : \mathcal{B} \rightarrow \mathbb{k}$ there is a conformal derivation $\alpha_f \in \text{cder } \mathfrak{F}$ such that $N(\alpha_f, \mathcal{B}) = 1$ and $\alpha_f(0)b = f(b)b$ for every $b \in \mathcal{B}$ (see Lemma 2(a)). Let $R = \mathbb{k}[\alpha_f(n) \mid f \in \mathcal{B}^*, n \in \mathbb{Z}_+]$ be the commutative polynomial algebra generated by all $\alpha_f(n)$. Lemma 2(b) states that R acts on both \mathfrak{F} and V_Λ and these actions agree with φ .

Fix an element $a \in \mathcal{B}$ and let $f : \mathcal{B} \rightarrow \mathbb{k}$ be defined by $f(b) = \delta_{a,b}$. Denote the corresponding conformal derivation by $a^\vee = \alpha_f \in \text{cder } \mathfrak{F}$. The algebra R is graded by $\Lambda_+ \oplus \mathbb{Z}$ by setting $\text{wt } a^\vee(n) = a$, $\deg a^\vee(n) = -n$.

Our next observation reduces the problem to the following claim.

Claim A. *For every homogeneous non-trivial linear combination x of the elements of \mathcal{T} such that $\text{wt } x = \lambda$ and $\deg x > \deg w_{\min}(\lambda) = \frac{1}{2}(\lambda|\lambda)$, there is $r \in R$, such that $\deg r = \deg w_{\min}(\lambda) - \deg x$ and $rx = c w_{\min}(\lambda)$ for some constant $0 \neq c \in \mathbb{k}$.*

Here $w_{\min}(\lambda)$ is given by (14).

Indeed, since φ is homogeneous, it is enough to show that $\varphi(x) \neq 0$ for a homogeneous linear combination x of \mathcal{T} . If $x = w_{\min}(\lambda)$, then $\varphi(x) = \pm v_\lambda \neq 0$ by Proposition 1. Otherwise, using Lemma 2(b) we get $r\varphi(x) = k\varphi(w_{\min}(\lambda)) \neq 0$, hence $\varphi(x) \neq 0$.

Step 3. Let $w = a_1(m_1) \cdots a_k(m_k) \in \mathcal{X}^*$ be a word in \mathcal{X} and consider a monomial $p = a_1^\vee(n_1) \cdots a_k^\vee(n_k) \in R$. Let $G < S_k$ be a subgroup of the group of permutations of k elements consisting of permutations that fix the k -tuple (a_1, \dots, a_k) , and let $H < G$ be the subgroup of G consisting of permutations that in addition fix (n_1, \dots, n_k) . We show that Claim A follows from

Claim B. *There exists an element $r_p \in R$ such that*

$$r_p w = \frac{1}{|H|} \sum_{\sigma \in G} a_1(m_1 + n_{\sigma(1)}) \cdots a_k(m_k + n_{\sigma(k)}). \quad (22)$$

Note that every word that appears in the right-hand side of (22) has coefficient 1.

Indeed, suppose Claim B is true. Define an order on \mathbb{Z}^k by letting $(l_1, \dots, l_k) > (l'_1, \dots, l'_k)$ if there is $1 \leq j \leq k$ such that $l_i = l'_i$ for $j+1 \leq i \leq k$, but $l_j > l'_j$.

Suppose that $w \in \mathcal{T}$, and let $(n_1, \dots, n_k) = \eta(w) \in \mathbb{Z}^k$ be given by (15). Recall that $n_1 \geq n_2 \geq \dots \geq n_k$ and if $n_i = n_{i+1}$ then $a_i \leq a_{i+1}$. By Proposition 1 we have $\rho(a_1(m_1 + n_1) \cdots a_k(m_k + n_k)) = w_{\min}(\text{wt } w) \neq 0$. Since \mathcal{T} linearly spans \mathfrak{F} , every word of weight $\text{wt } w$ and degree less than $\deg w$ is 0 (in other words, Proposition 2 holds). Therefore, if $(n'_1, \dots, n'_k) > (n_1, \dots, n_k)$ then $\rho(a_1(m_1 + n'_1) \cdots a_k(m_k + n'_k)) = 0$.

If $\sigma \in G \setminus H$, then $(n_1, \dots, n_k) < (n_{\sigma(1)}, \dots, n_{\sigma(k)})$. Therefore, (22) implies that

$$r_p \rho(w) = \rho(a_1(m_1 + n_1) \cdots a_k(m_k + n_k)) \neq 0.$$

This proves Claim A for $x = w \in \mathcal{T}$.

Take now another word $u \in \mathcal{T}$ such that $u \neq w$, $\text{wt } u = \text{wt } w$, $\deg u = \deg w$. Let $\mu \in S_k$ be a permutation such that $u = a_{\mu(1)}(m'_1) \cdots a_{\mu(k)}(m'_k)$. Set $(n'_1, \dots, n'_k) = \eta(u) \in \mathbb{Z}^k$. Suppose that $(n'_1, \dots, n'_k) \leq (n_1, \dots, n_k)$. Then we claim that $r_{p(w)} \rho(u) = 0$. Indeed, we have

$$r_{p(w)} u = \sum_{\sigma \in \mu G \mu^{-1}} a_{\mu(1)}(m'_1 + n_{\sigma(1)}) \cdots a_{\mu(k)}(m'_k + n_{\sigma(k)}).$$

This belongs to $\text{Ker } \rho$ if we show that $(n'_1, \dots, n'_k) < (n_{\sigma(1)}, \dots, n_{\sigma(k)})$ for all $\sigma \in \mu G \mu^{-1}$. If there were an equality in $(n'_1, \dots, n'_k) \leq (n_1, \dots, n_k) \leq (n_{\sigma(1)}, \dots, n_{\sigma(k)})$, then $\eta(u)$ would define the same colored partition of $\deg w - \deg_{\min}(\text{wt } w)$ as does $\eta(w)$, and this contradicts the assumption $u \neq w \in \mathcal{T}$, see §5.

Now we can prove Claim A. Let $x \in \mathbb{k}\langle \mathcal{X} \rangle$ be a homogeneous linear combination of words in \mathcal{T} . To every word $u = a_1(m_1) \cdots a_k(m_k) \in \mathcal{T}$ that appears in this combination we correspond the sequence $(n_1, \dots, n_k) = \eta(u) \in \mathbb{Z}^k$, given by (15). All these sequences are pairwise different, since the words in x are all of the same weight and degree. Let w be the word that yields the maximal sequence. Take $p = p(w) \in R$ as above, and let r_p be as in Claim B. Then the above argument shows that $0 \neq r_p w \in \mathbb{k}w_{\min}(\text{wt } w)$ and $r_p u = 0$ for every other word $u \in \mathcal{T}$ that appears in x .

Step 4. It remains to construct for a word $w = a_1(m_1) \cdots a_k(m_k) \in \mathcal{X}^*$ and for a monomial $p = a_1^\vee(n_1) \cdots a_k^\vee(n_k) \in R$ an element $r_p \in R$ such that (22) holds. Denote the right-hand side of (22) by $S_p(w)$.

Let us introduce a partial order on the set of monomials in R . Suppose $a_i = a_j = a$ for some $1 \leq i \neq j \leq k$, and consider the result of substituting $a^\vee(n_i) a^\vee(n_j)$ by $a^\vee(n_i + n_j) a(0)$ in p . We write $q \prec p$ if q can be obtained by a number of such substitutions.

Let us now calculate the action of p on w . Recall that R acts on $\mathbb{k}\langle \mathcal{X} \rangle$ by derivations, so that $a_i(n)^\vee(a_j(m)) = \delta_{i,j} a_j(m + n)$. For $1 \leq j \leq k$ denote $\Theta_j = \{i \mid a_i = a_j\} \subset \{1, \dots, k\}$. Then

$$pw = \sum_{i_1 \in \Theta_1, \dots, i_k \in \Theta_k} a_1\left(m_1 + \sum_{j=1}^k \delta_{i_j,1} n_j\right) \cdots a_k\left(m_k + \sum_{j=1}^k \delta_{i_j,k} n_j\right).$$

The crucial observation is that

$$pw = \sum_{q \preccurlyeq p} c(p, q) S_q(w)$$

for some positive integers $c(p, q)$. Using this, we construct r_p by induction on the number $l = l(p) = \#\{1 \leq j \leq k \mid n_j \neq 0\}$. If $l = 0$ or 1 , then $q \preccurlyeq p$ implies $q = p$ and we take $r_p = c(p, p)^{-1}p$. Otherwise, set

$$r_p = c(p, p)^{-1} \left(p - \sum_{q \not\preccurlyeq p} c(p, q) r_q \right).$$

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